Math Circle

Basic Complex Numbers

> ***Note:*** *Feel free to skip over any concept you already know, but still do the proofs.*

Introduction to Real Numbers

$\mathbb{R}$ denotes the set of real numbers.

Examples of real numbers: -1, 0, 2, $\sqrt{2}$, 3/4, π, e

The Problem with Real Numbers

Some polynomials do not have real solutions. For example, consider the polynomial:

$$\mathbf{x^2 + 1 = 0}$$

To solve this problem, we simply have to introduce a new variable, the definition of which is that it satisfies the polynomial $x^2 + 1 = 0$. This is actually called *adjoining an element to a field* in abstract algebra, and is completely natural.

We usually let this number be denoted as **i**, where $\mathbf{i^2 = -1}$.

Complex Number Definition

Definition: A number of the form **a + bi** is called a complex number if a and b are real numbers.

Let z = a + bi. Then:

- **Re(z)** denotes the real part of z, which is a
- **Im(z)** denotes the imaginary part of z, which is b

Examples of Real and Imaginary Parts:

• For z = 5 + 3i: Re(z) = 5, Im(z) = 3

• For z = -2 + 7i: Re(z) = -2, Im(z) = 7

• For z = 4 - 6i: Re(z) = 4, Im(z) = -6

• For z = 9: Re(z) = 9, Im(z) = 0

• For z = -3i: Re(z) = 0, Im(z) = -3

Addition of Complex Numbers

Example:

(3 + 2i) + (1 + 5i) = (3 + 1) + (2 + 5)i = 4 + 7i

Exercise 1 - Addition Practice:

a) (2 + 3i) + (4 + i) = _____

b) (7 + 2i) + (-3 + 4i) = _____

c) (5 - 6i) + (2 + 8i) = _____

d) (-1 + 3i) + (6 - 7i) = _____

General Expression: (a + bi) + (c + di) = _____

Multiplication of Complex Numbers

HINT: Use FOIL (First, Outer, Inner, Last)

Example:

(2 + 3i)(1 + 4i) = 2(1) + 2(4i) + 3i(1) + 3i(4i)

= 2 + 8i + 3i + 12i²

= 2 + 11i + 12(-1) = 2 + 11i - 12 = -10 + 11i

Exercise 2 - Multiplication Practice:

a) (1 + 2i)(3 + i) = _____

b) (4 + i)(2 - 3i) = _____

c) (3 - 2i)(1 + 5i) = _____

d) (2 + 4i)(3 - i) = _____

General Expression: (a + bi)(c + di) = _____

Powers of i

Exercise 3 - Powers of i:

Power	Value
i^0	
i^1	
i^2	
i^3	
i^4	
i^5	
i^6	

Question: Find the smallest positive integer k such that i^k = 1.

Answer: k = _____

Evaluating Polynomials with i

Exercise 4 - Evaluating Polynomials with i:

a) $2 + 3i + 4i^2$ = _____

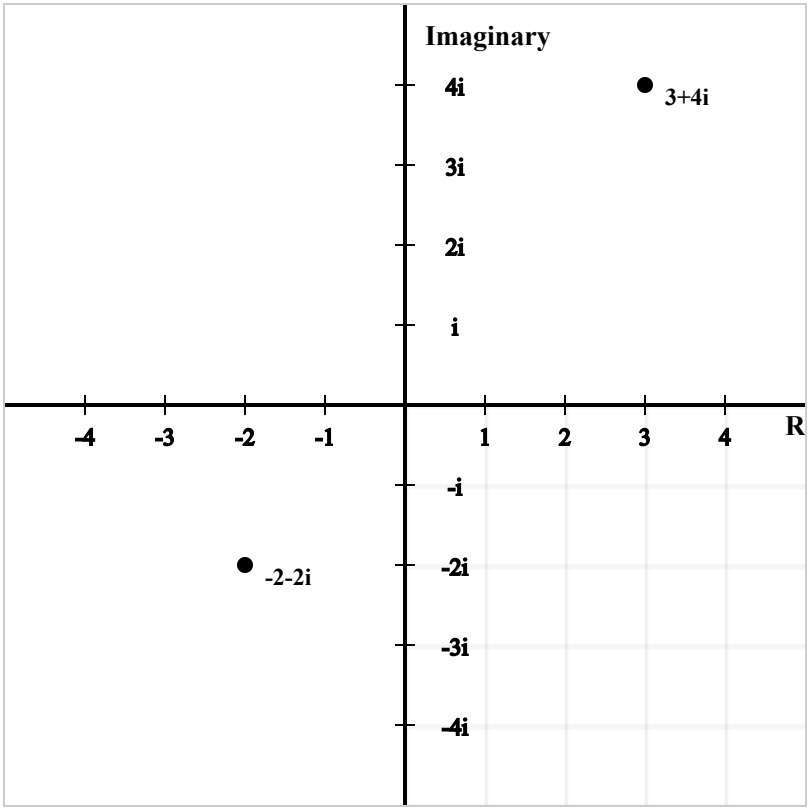
b) $1 + i + i^2 + i^3$ = _____

c) $5 - 2i + 3i^2 + i^3$ = _____

d) $i^5 + i^6 + i^7 + i^8$ = _____

The Complex Plane

The complex plane is just like the xy-plane, except that the x-component of z is Re(z), and the y-component of z is Im(z).



Absolute Value and Complex Conjugate

Absolute Value: For $z = a + bi$, $|z| = \sqrt{(a^2 + b^2)}$

Complex Conjugate: For $z = a + bi$, $\bar{z} = a - bi$

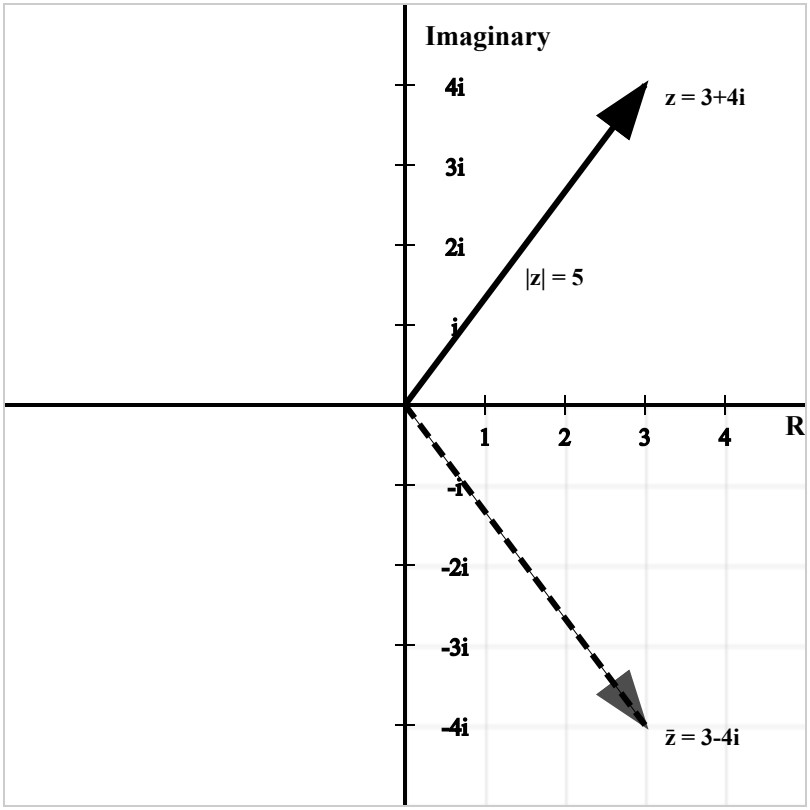
Conceptual Understanding: The absolute value $|z|$ represents the distance from the origin (0, 0) to the point z in the complex plane. It's like finding the hypotenuse of a right triangle where the legs have lengths $|a|$ and $|b|$.

Example with z = 3 + 4i:

$|3 + 4i| = \sqrt{(3^2 + 4^2)} = \sqrt{(9 + 16)} = \sqrt{25} = 5$

3 + 4i conjugate = 3 - 4i

Visual Representation of $|z|$ and $\bar{z}$ for $z = 3 + 4i$



Exercise 5 - Calculate |z| and z̄ for:

a) z = 5 + 12i

|z| = _____ , z̄ = _____

b) z = -3 + 4i

|z| = _____ , z̄ = _____

c) z = 1 - 7i

|z| = _____ , z̄ = _____

d) z = -2 - 6i

|z| = _____ , z̄ = _____

Challenge Questions

The following questions require writing mathematical proofs. Show all your work and reasoning clearly.

Challenge Question A:

Statement: The definition of complex numbers are numbers that can be expressed as a+bi if a and b are real. Use this definition to prove that a+bi is real even for a and b as all complex numbers.

Write your proof here:

Challenge Question B:

Statement: Prove that $a_0 + a_1 \cdot i + a_2 \cdot i^2 + a_3 \cdot i^3 + ... + a_n \cdot i^n$ is complex (or equivalently in sigma notation: $\sum_{k=0}^{n} a_k \cdot i^k$), given that a_k is a real number.

Write your proof here:

Challenge Question C:

Statement: Prove the above result when a_k are complex numbers.

Write your proof here: