Binomial Theorem

What is the Binomial Theorem:

The Binomial Theorem tells us how to expand an expression like (a+b)^n without multiplying it out the long way.

- For small powers, we can expand directly:
  (a+b)^2 = a^2 + 2ab + b^2
  (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3

- The coefficients follow a pattern given by **Pascal's Triangle** or **combinations** (nCk).

- The general formula is:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

x

Pascal's Triangle:

Each number is the sum of the two numbers above it.

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1
\end{array}$$

These are the coefficients in the expansion.

Combinations Connection:

`nCk = n! / (k!(n-k)!)`

This tells us how many ways to choose k items from n.

It also gives the coefficient of the a^(n-k)b^k term in the expansion.

Warmup:

1. Expand $(x+1)^2$.

2. Expand $(x+1)^3$.

3. What do you notice about the coefficients?

Practice Problem 1:

Expand $(x+y)^4$.

Explanation:
Use Pascal's Triangle row 4: 1,4,6,4,1

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Practice Problem 2:

Find the coefficient of x^3y^2 in $(x+y)^5$.

Explanation:
We want the term where $x^{(5-2)}y^2 = x^3y^2$.
Coefficient = $5C2 = 10$.
Answer: $10x^3y^2$.

Practice Problem 3:

Expand $(2x - 3)^3$.

Explanation:
Use row 3: 1, 3, 3, 1.
$(2x-3)^3 = (2x)^3 + 3(2x)^2(-3) + 3(2x)(-3)^2 + (-3)^3$
$= 8x^3 - 36x^2 + 54x - 27$

Try By Yourself Problems:

1. Expand $(x+y)^5$.
2. Find the coefficient of x^2y^3 in $(x+y)^5$.
3. Expand $(a-b)^4$.
4. Find the coefficient of a^7b^3 in $(a+b)^{10}$.
5. Expand $(3x+1)^2$.

Extra Time Problems:

Find the general term (the k-th term) in the expansion of $(2x - 3y)^8$.
How many terms are there in the expansion of $(a+b)^9$?
Which is/are the middle term(s)? Write them explicitly.
In the expansion of $(1+x)^{12}$, what is the largest coefficient?
Find the coefficient of x^5 in the expansion of $(2 + x)^8$.
Expand just enough of $(1+x)^{10}$ to find the sum of the coefficients.
Find the coefficient of x^4 in $(2 - x)^6$.
Use the binomial theorem to expand $(1+x)^5(1-x)^5$ and simplify.
When flipping a fair coin 8 times, what is the probability of getting exactly 5 heads?
(Hint: relate to coefficients in $(H+T)^8$).

In the expansion of $(x + 1/x)^6$, find the term independent of x.
In the expansion of $(1+x)^n$, show that the sum of the squares of the coefficients is $C(2n, n)$.

Solutions:

1. $(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

2. Coefficient of x^2y^3 = 5C3 = 10

3. $(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$

4. Coefficient = 10C3 = 120

5. $(3x+1)^2 = 9x^2 + 6x + 1$

1. $T(k+1) = 8Ck * (2x)^{(8-k)} * (-3y)^k$

2. Total terms = 10. Middle term = 5th and 6th terms.

   - 5th: $126a^5b^4$

   - 6th: $126a^4b^5$

3. 12C6 = 924

4. 2016

5. 1024

6. -240

7. $(1+x)^5(1-x)^5 = (1 - x^2)^5$

8. 7/32

9. 20

10. C(2n, n)