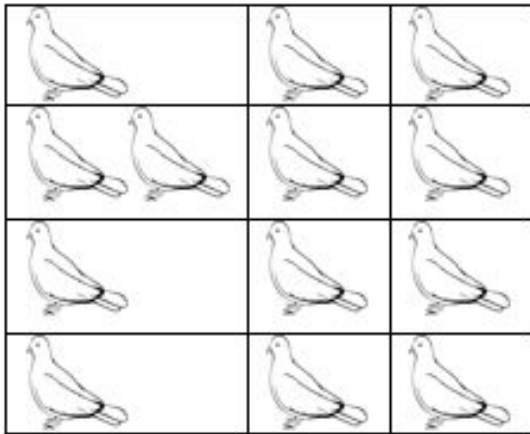


Pigeonhole Principle

Definition:

The Pigeonhole Principle states:

If you have more objects (pigeons) than containers (pigeonholes), then at least one container must hold more than one object.



Examples:

- 13 socks in 12 drawers $\rightarrow$ at least one drawer has ≥ 2 socks.
- 367 people $\rightarrow$ at least two share the same birthday.

Key Idea:

- Count objects and containers; if objects $>$ containers, overlap is guaranteed.

Statement:

If n objects are placed into k containers, at least one container contains at least $\lceil n/k \rceil$ objects.

Example:

- 100 students $\rightarrow$ 9 classrooms $\rightarrow$ at least one classroom has $\lceil 100/9 \rceil = 12$ students.

1. Identify the objects (pigeons).
2. Identify the containers (pigeonholes).
3. Decide whether basic or generalized principle applies.
4. Consider the worst-case or maximum spread for tricky problems.
5. Use modular arithmetic, parity, geometry, or combinatorics when objects/containers aren't obvious.

Examples

Example 1:

- 5 pencils, 4 boxes $\rightarrow \lceil 5/4 \rceil = 2 \rightarrow$ at least one box has 2 pencils.

Example 2:

- 31 students, 12 months $\rightarrow \lceil 31/12 \rceil = 3 \rightarrow$ at least one month has 3 birthdays.

Example 3:

- 7 integers $\rightarrow$ modulo 6 $\rightarrow$ 7 numbers, 6 remainders $\rightarrow$ two numbers share a remainder $\rightarrow$ difference divisible by 6.

Example 4:

Let S be any set of 20 distinct positive integers less than or equal to 100. Prove that there exist two disjoint nonempty subsets of S with the same sum.

First, note that S has $2^{20} - 1 = 1,048,575$ nonempty subsets.

The largest possible sum of any subset is at most $100 + 99 + \dots + 81 =$

1910.

Since there are far more subsets than possible sums, by the pigeonhole principle, at least two subsets must have the same sum.

Because there are so many subsets with the same sum, we can pick two that do not share any elements.

Therefore, there exist two disjoint nonempty subsets with equal sum.

Practice Problems

1. You place 9 coins into 4 boxes. Show that at least one box contains at least 3 coins.
2. A drawer contains 10 red socks, 12 blue socks, and 8 green socks. How many socks must you pick to guarantee at least 3 of the same color?
3. In a group of 50 people, show that at least 5 were born in the same month.
4. Place 23 students into 5 teams. What is the minimum number of students in the largest team?
5. Show that in any set of 7 integers, there are at least two whose difference is divisible by

Challenge problems

6. Fifty-one points are placed in a square of side length 1. Prove that there is a circle of radius $1/7$ that contains three of the points.
7. Let S be any set of 20 distinct positive integers ≤ 100 . Show that there exist two disjoint nonempty subsets of S with the same sum.
8. Consider $2n+1$ integers. Prove that there exists a subset of $n+1$ integers whose sum is divisible by n .
9. Let A be a set of 101 integers from 1 to 200. Prove that there exist two distinct subsets of A with the same sum.

Solutions:

1. $\lceil 9/4 \rceil = 3 \rightarrow$ at least one box has 3 coins.
2. Worst case: pick 2 red + 2 blue + 2 green = 6 $\rightarrow$ next pick guarantees 3 $\rightarrow$ 7 socks.
3. 50 people, 12 months $\rightarrow \lceil 50/12 \rceil = 5 \rightarrow$ at least 5 share a birth month.
4. 23 students, 5 teams $\rightarrow \lceil 23/5 \rceil = 5 \rightarrow$ largest team has ≥ 5 students.
5. Numbers modulo 6 $\rightarrow$ 7 numbers, 6 remainders $\rightarrow$ at least 2 share remainder $\rightarrow$ difference divisible by 6.
6. Divide the square into 49 smaller squares of side $1/7$. 51 points $\rightarrow$ one small square has ≥ 3 points $\rightarrow$ circle radius $1/7$ contains them.
7. Subset sums: $2^{20} > 100 \rightarrow$ by pigeonhole, two disjoint subsets have the same sum.
8. Consider sums modulo $n \rightarrow 2n+1$ integers $\rightarrow n+1$ numbers' sum divisible by n (pigeonhole on remainders).
9. Number of subsets = $2^{101} >$ sum range $\rightarrow$ by pigeonhole, two subsets have the same sum.