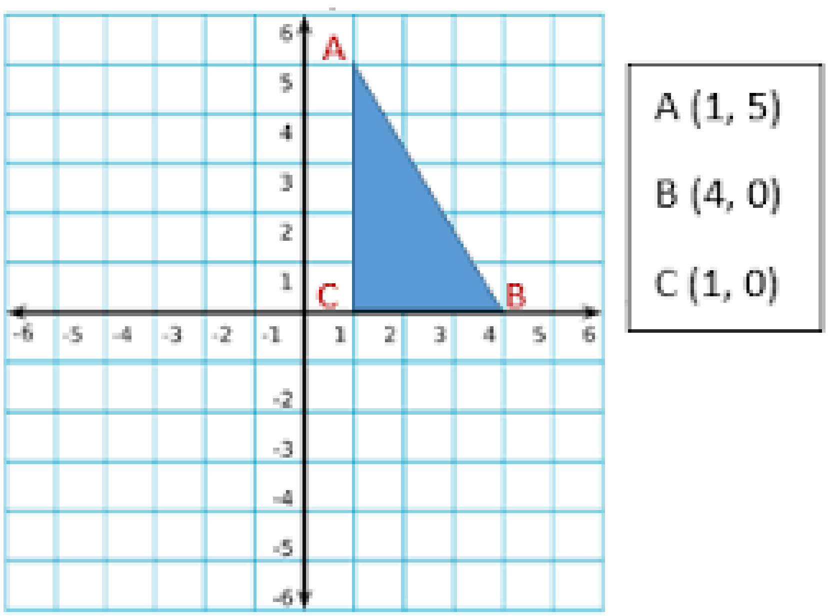
Math Circle

Coordinate Geometry



Key Equations

1. **Slope Relationships**
    - Parallel lines → same slope.
    - Perpendicular lines → slopes multiply to -1 of the reciprocal.
2. **Equation of a Line**
    - Slope–intercept form: $y = mx + b$
    - Point–slope form: $y - y_1 = m(x - x_1)$
3. **Circle Equation**
    - $(x - h)^2 + (y - k)^2 = r^2$

- To find radius or center from general form $Ax^2 + Ay^2 + Dx + Ey + F = 0$, complete the square.

4. **Distance Between a Point and a Line**

   - For point (x_1, y_1) and line $Ax + By + C = 0$:
     Distance = $|Ax_1 + By_1 + C| / \sqrt{(A^2 + B^2)}$

5. **Area of a Triangle Using Coordinates**

   - For (x_1, y_1), (x_2, y_2), (x_3, y_3):
     Area = $\frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$

Warmup

1. Find the area of the triangle with vertices (1, 2), (4, 6), and (7, 2).
2. What is the distance from (3, –1) to the line $3x - 4y - 12 = 0$?
3. Find the equation of a circle centered at (2, –3) passing through (6, 5).

Practice Problem 1

Problem:
The line passing through (2, 3) and (8, y) is perpendicular to the line $3x + 2y = 7$. Find y.

Explanation:
Slope of $3x + 2y = 7 \rightarrow y = -(3/2)x + 7/2 \rightarrow$ slope = –3/2.
Perpendicular slope = 2/3.
Slope of points (2, 3) and (8, y) = $(y - 3)/(8 - 2) = (y - 3)/6 = 2/3 \rightarrow y - 3 = 4 \rightarrow y = 7$.
Answer: y = 7

Practice Problem 2

Problem:
Find the coordinates of the point on the line $2x + y = 5$ that is closest to (3, 4).

Explanation:
Closest point lies on the perpendicular from (3, 4).
Line: $2x + y = 5 \rightarrow$ slope = $-2 \rightarrow$ perpendicular slope = ½.
Equation through (3, 4): $y - 4 = \frac{1}{2}(x - 3) \rightarrow y = \frac{1}{2}x + 5/2$.
Solve system:
$2x + y = 5 \rightarrow y = 5 - 2x$.
Set equal: $\frac{1}{2}x + 5/2 = 5 - 2x \rightarrow$ multiply by 2: $x + 5 = 10 - 4x \rightarrow 5x = 5 \rightarrow x = 1$.
Then $y = 5 - 2(1) = 3$.
Answer: (1, 3)

Practice Problem 3

Problem:
A circle passes through A(2, 3) and B(6, 7), and its center lies on the line $y = x$. Find the equation of the circle.

Explanation:
Let center = (h, h).
Radius² = $(h - 2)^2 + (h - 3)^2 = (h - 6)^2 + (h - 7)^2$ (since both on circle).
Simplify: $2h^2 - 10h + 13 = 2h^2 - 26h + 85 \rightarrow 16h = 72 \rightarrow h = 4.5$.
Center = (4.5, 4.5).
Use A(2, 3): $r^2 = (4.5 - 2)^2 + (4.5 - 3)^2 = (2.5)^2 + (1.5)^2 = 8.5$.
Answer: $(x - 4.5)^2 + (y - 4.5)^2 = 8.5$

Practice Problem 4

Problem:
Find the area of the quadrilateral with vertices A(1, 1), B(5, 1), C(4, 4), and D(0, 4).

Explanation:
Split into triangles: A–B–C and A–C–D.
Area(ABC) = $\frac{1}{2}|1(1-4) + 5(4-1) + 4(1-1)| = \frac{1}{2}|-3 + 15| = 6$.
Area(ACD) = $\frac{1}{2}|1(4-4) + 4(4-1) + 0(1-4)| = \frac{1}{2}|0 + 12 + 0| = 6$.
Total = 12.
Answer: 12

Practice Problem 5

Problem:
Find the equation of the line that divides the segment joining A(–2, 3) and B(4, –1) in the ratio 1:2 (internally).

Explanation:
Section formula:
$x = (1×4 + 2×(–2)) / (1 + 2) = (4 – 4)/3 = 0$.
$y = (1×(–1) + 2×3) / (3) = (–1 + 6)/3 = 5/3$.
Point dividing the line: (0, 5/3).
Line AB: slope = $(–1 – 3)/(4 – (–2)) = –4/6 = –2/3$.
Equation: $y – 5/3 = –2/3(x – 0) \rightarrow y = –2/3x + 5/3$.
Answer: $y = –(2/3)x + 5/3$

Challenge Problems

1. A line passes through (1, 2) and cuts the circle $x^2 + y^2 = 25$ at points P and Q. If PQ = 10, find the slope of the line.

2. Find all points (x, y) that satisfy both $x^2 + y^2 = 25$ and $x + y = 7$.

3. The vertices of a triangle are A(–1, 2), B(3, 4), and C(2, –3). Find the equation of the altitude from A.

4. A line is tangent to the circle $(x – 3)^2 + (y – 2)^2 = 13$ at point (6, 5). Find its equation.

5. The points A(1, –1), B(4, 5), and C(k, 7) are collinear. Find k.

6. Find the locus of a point that moves so that its distance from (3, 0) is twice its distance from the y-axis.

7. Find the area of the triangle formed by the lines $x + y = 4$, $x = 0$, and $y = 0$.

Solutions

1. Using chord formula → slope = ±3/4

2. Substitute $y = 7 – x \rightarrow x^2 + (7 – x)^2 = 25 \rightarrow 2x^2 – 14x + 24 = 0 \rightarrow x = 3, 4$ → Points (3, 4), (4, 3)

3. Slope of BC = $(-3 - 4)/(2 - 3) = -7/-1 = 7$ → perpendicular slope = $-1/7$ → Equation: $y - 2 = -1/7(x + 1)$

4. Radius slope = $(5 - 2)/(6 - 3) = 1$ → tangent slope = -1 → Equation: $y - 5 = -1(x - 6)$ → $y = -x + 11$

5. Slope AB = $(5 - (-1))/(4 - 1) = 6/3 = 2$ → $(7 - 5)/(k - 4) = 2$ → $2/(k - 4) = 2$ → $k = 5$

6. Let $P(x, y)$. Distance from $(3, 0)$: $\sqrt{(x - 3)^2 + y^2}$. Distance from y-axis: $|x|$. Equation: $\sqrt{(x - 3)^2 + y^2} = 2|x|$ → $(x - 3)^2 + y^2 = 4x^2$. Simplify: $3x^2 + 6x + y^2 - 9 = 0$.

7. Intercepts form right triangle with legs 4 and 4 → Area = $\frac{1}{2}(4\times4) = 8$